Analyses of Policy Gradient for Language Model Finetuning and Optimal Control

Noam Razin
Tel Aviv University

MML Seminar MPI MIS + UCLA 7 March 2024

Optimization and Generalization in Modern Machine Learning

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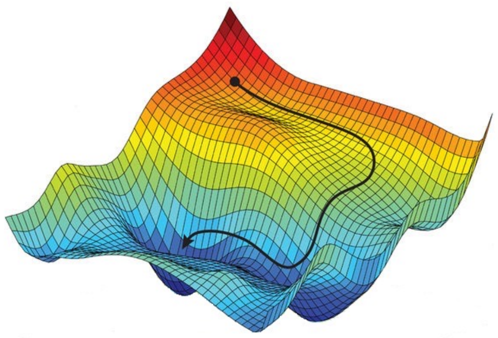
Optimization



Minimize a **non-convex** training objective

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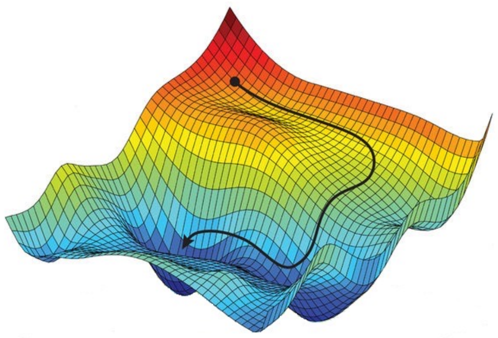

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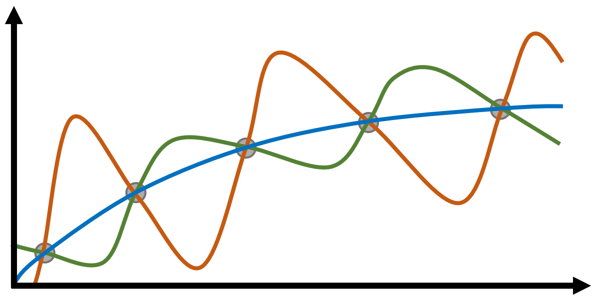

Performance on **data unseen in training**

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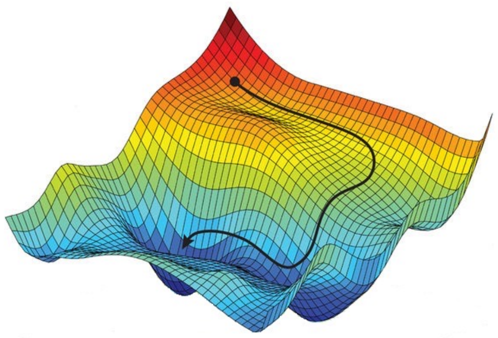

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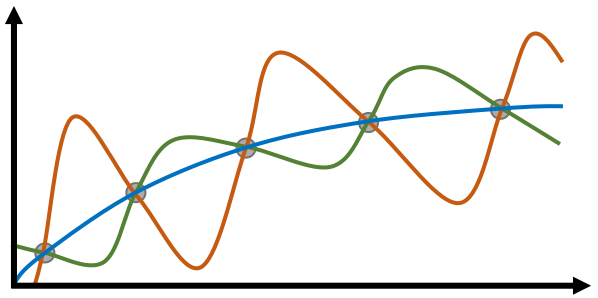

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Determined by **implicit bias** of training algorithm

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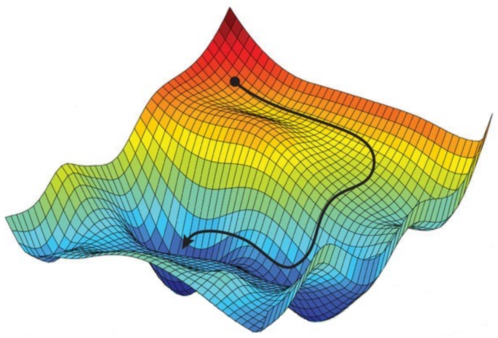

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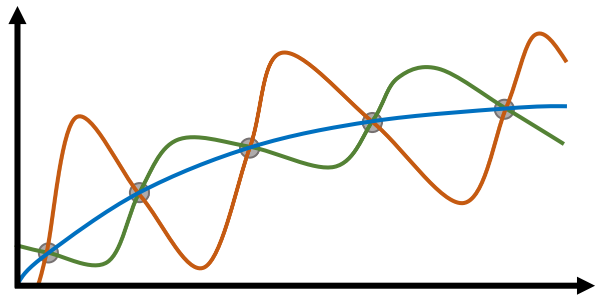

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Gradient-based methods are the workhorse behind
optimization and generalization in modern machine learning

Supervised Learning vs Optimal Control/Reinforcement Learning

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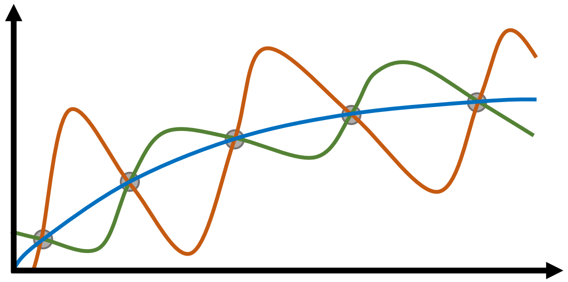
Supervised Learning



Task: Learn predictor minimizing loss over **labeled data**

Training Algorithm: Gradient descent

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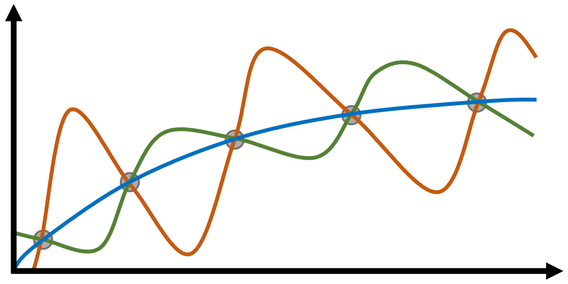

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Optimization Dynamics and Implicit Bias

Extensively Studied

(e.g., Neyshabur et al. 2014, Gunasekar et al. 2017, Soudry et al. 2018, Arora et al. 2019, Ji & Telgarsky 2019; **R** et al. 2020/21/22, Pesme et al. 2021, Lyu et al. 2021, Boursier et al. 2022, Andriushchenko et al. 2023, Frei et al. 2023, Jin & Montúfar 2023, Abbe et al. 2023)

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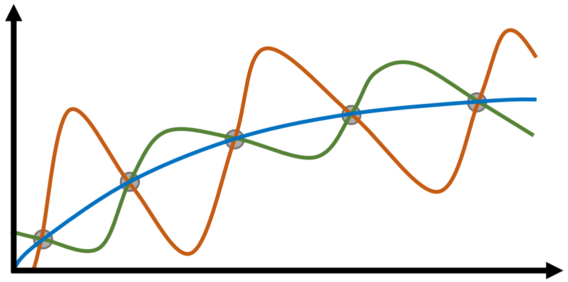

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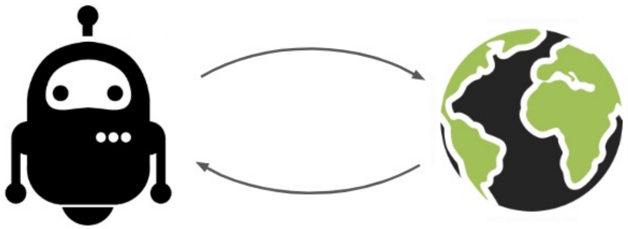

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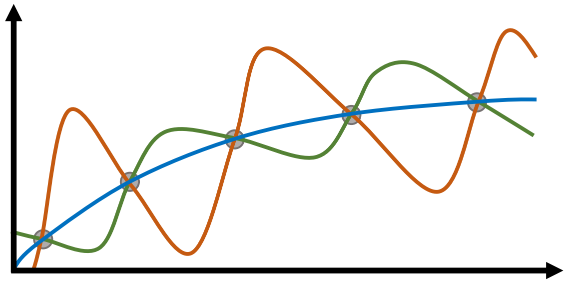

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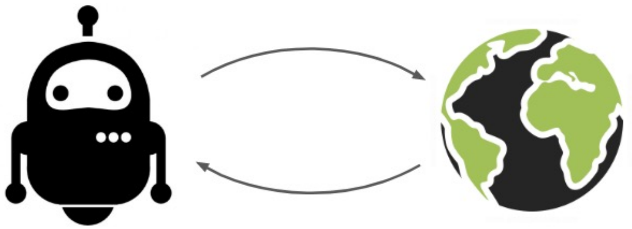

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Limited Understanding

(Fazel et al. 2018, Mei et al. 2020, Hu et al. 2021)

Sources

Optimization

Vanishing Gradients in Reinforcement Finetuning of Language Models

R + Zhou + Saremi + Thilak + Bradley + Nakkiran + Susskind + Littwin | *ICLR 2024*

Implicit Bias

Implicit Bias of Policy Gradient in Linear Quadratic Control: Extrapolation to Unseen Initial States

R + Alexander + Cohen-Karlik + Giryes + Globerson + Cohen | *arXiv 2024*

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🤖

LM p_θ

θ - parameters

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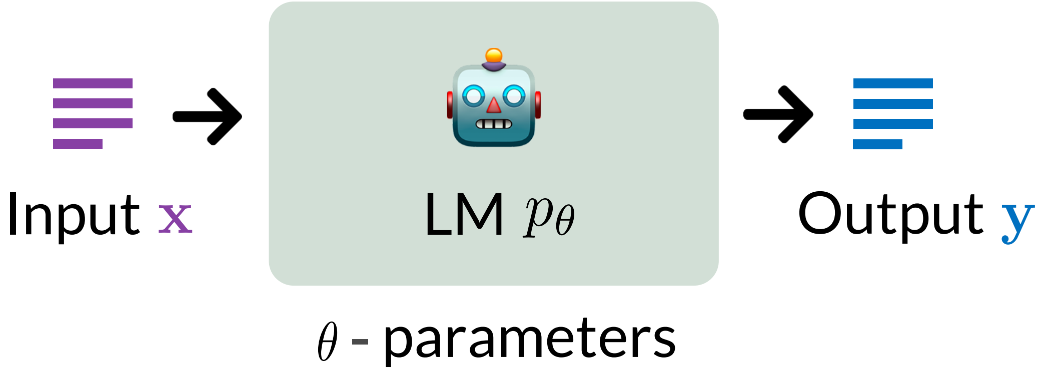

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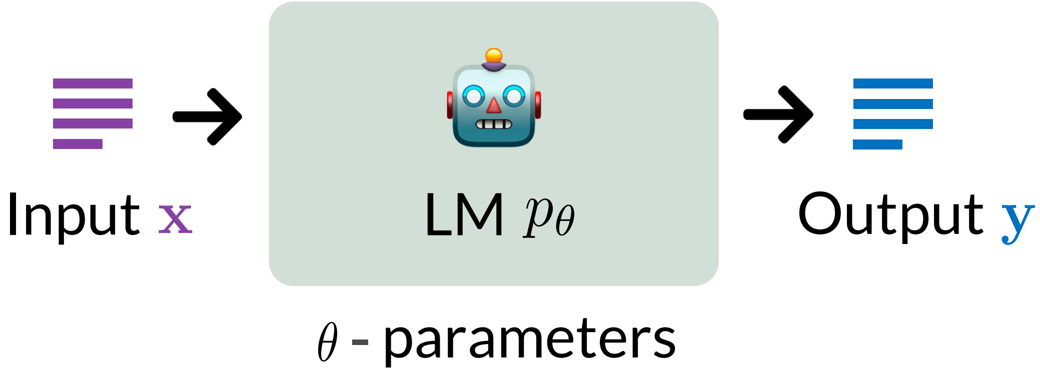

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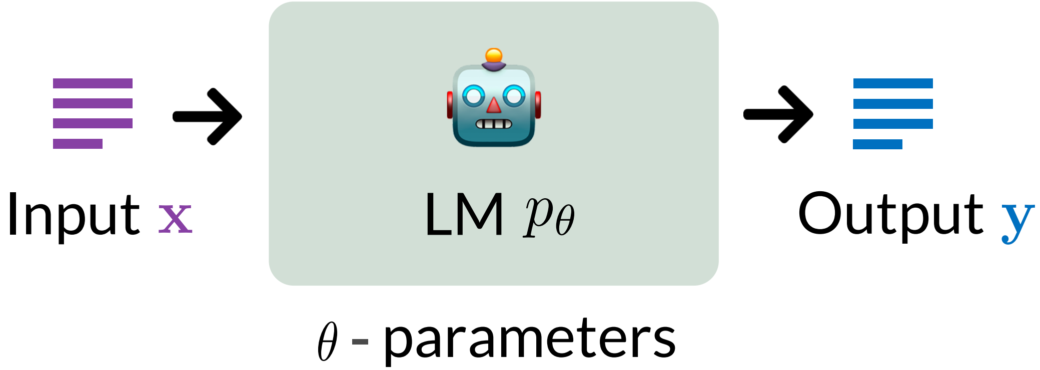

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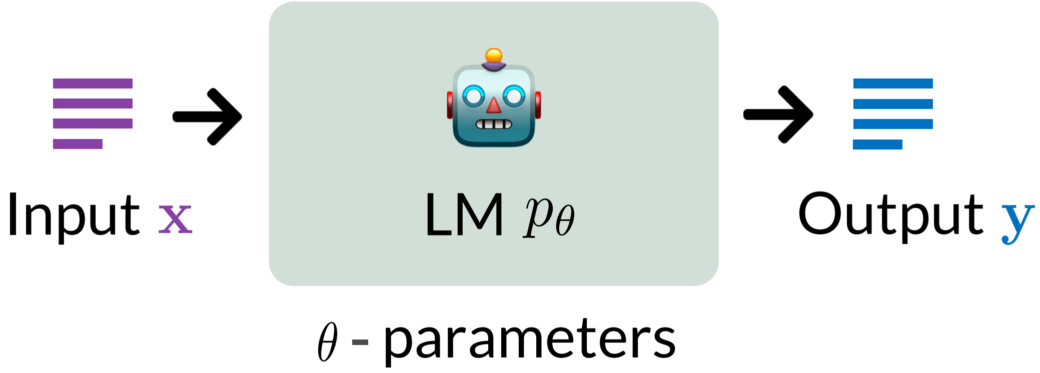

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softmax is used for producing next-token probabilities

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Limitations:

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- Labeled data is expensive

Reinforcement Finetuning of LMs

Limitations of SFT led to wide adoption of a **reinforcement learning**-based approach

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Can be problematic when finetuning text distribution differs from pretraining

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Finding I

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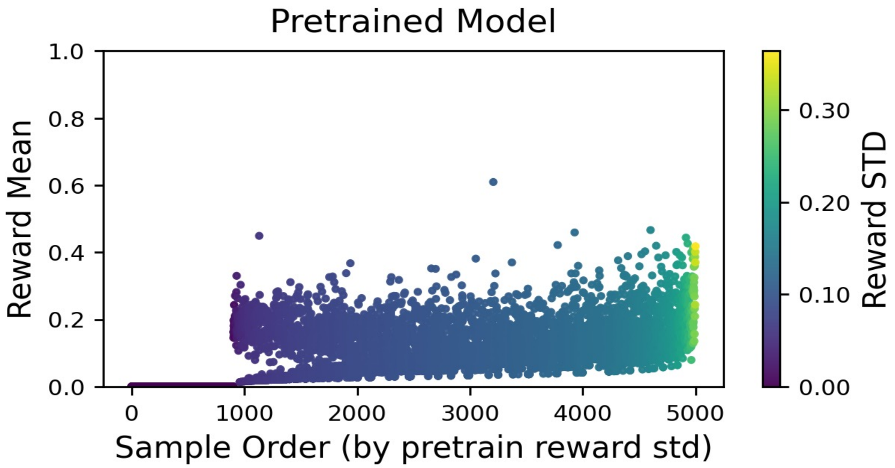

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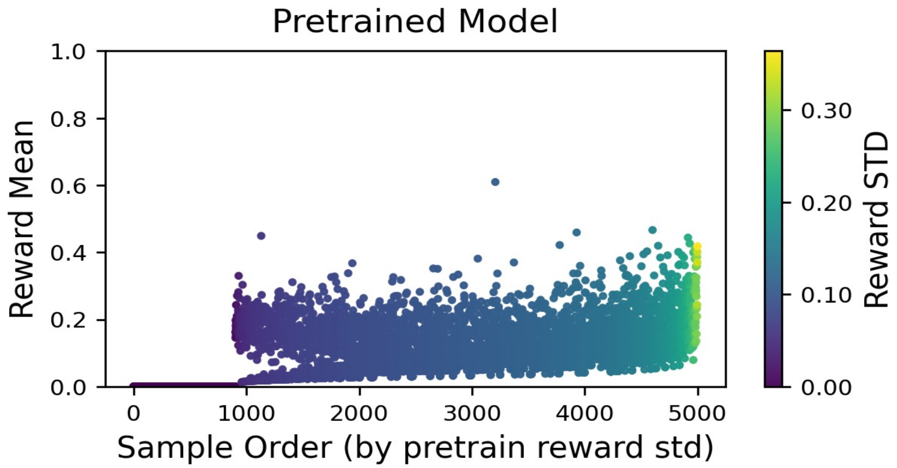

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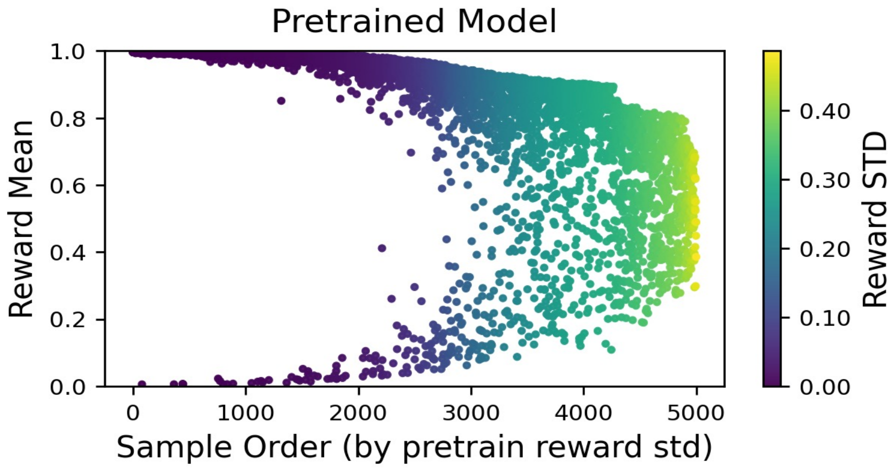

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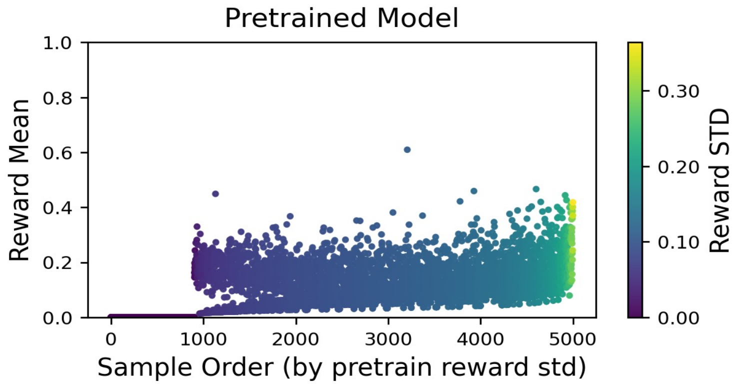

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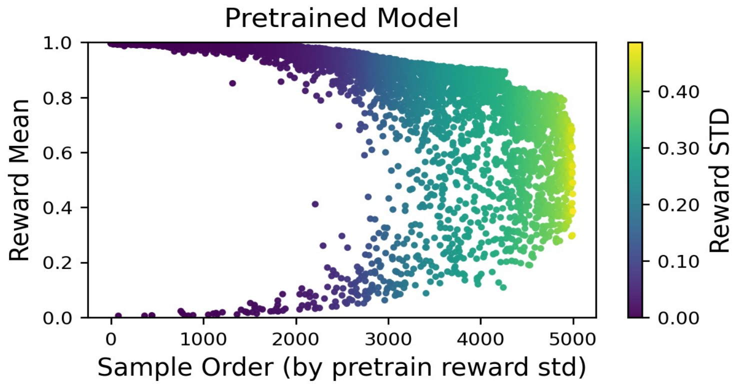

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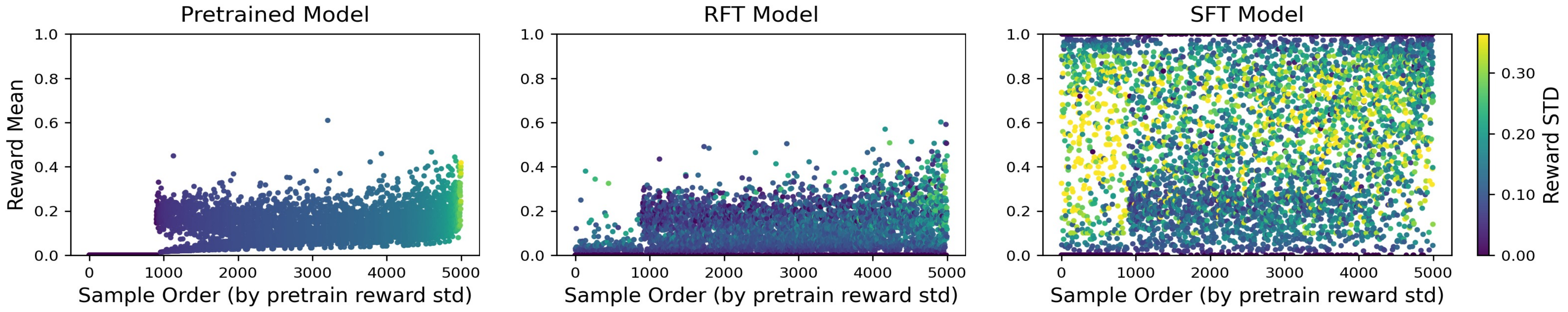

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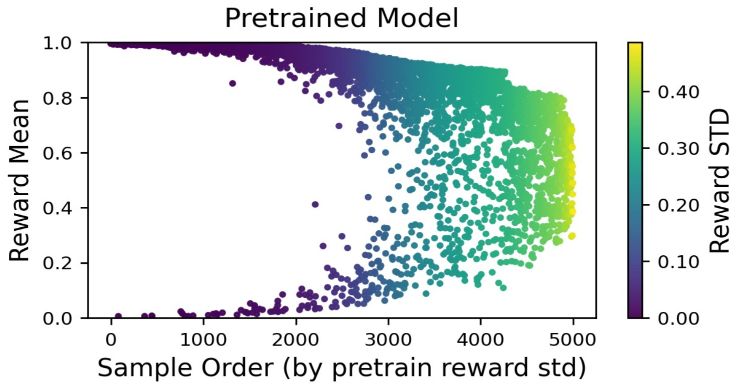

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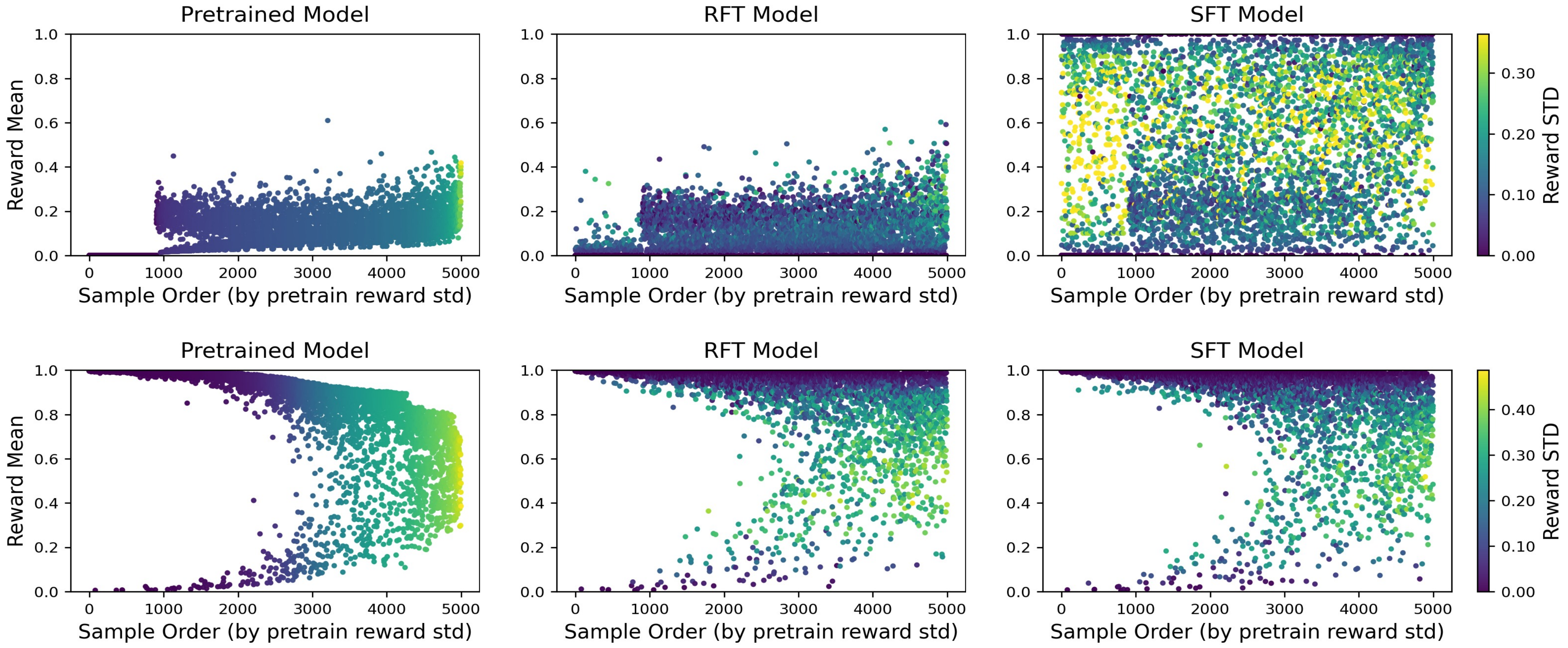

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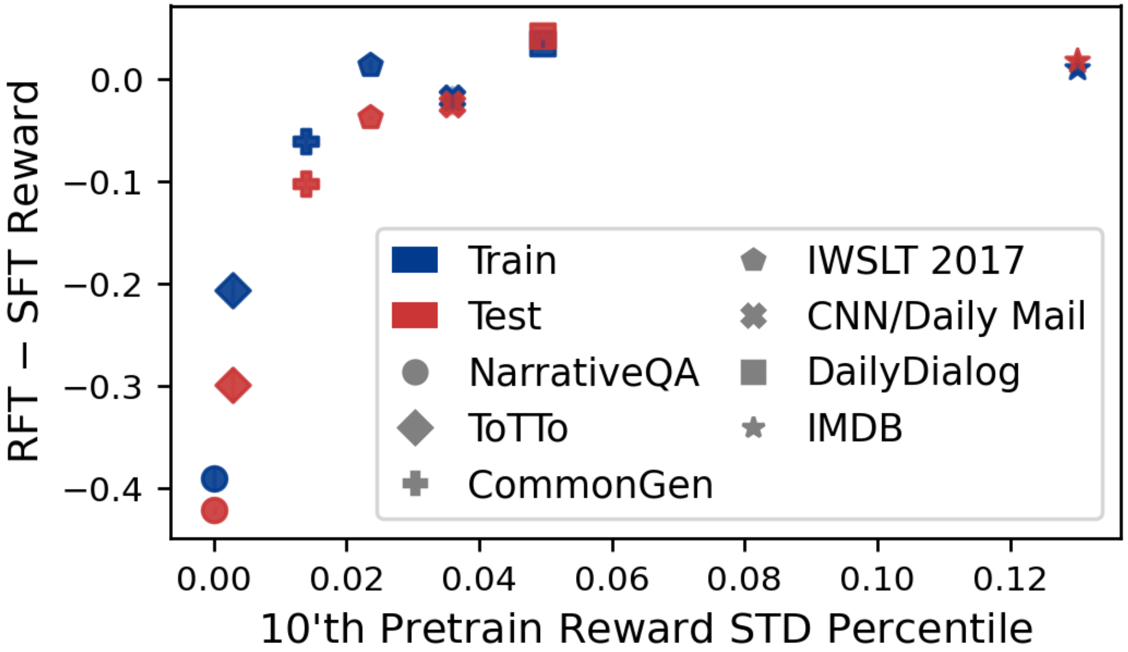

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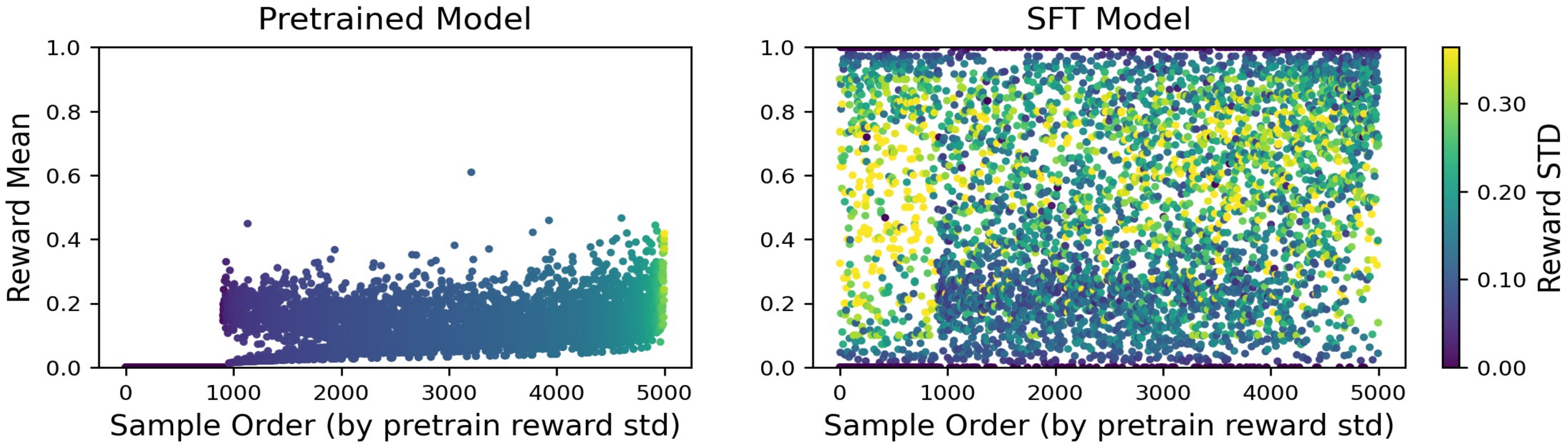
NarrativeQA (train)



Overcoming Vanishing Gradients in RFT

Common Heuristics: Increasing learning rate, temperature, entropy regularization ❌

Observation: Initial SFT phase reduces number of inputs with small reward std

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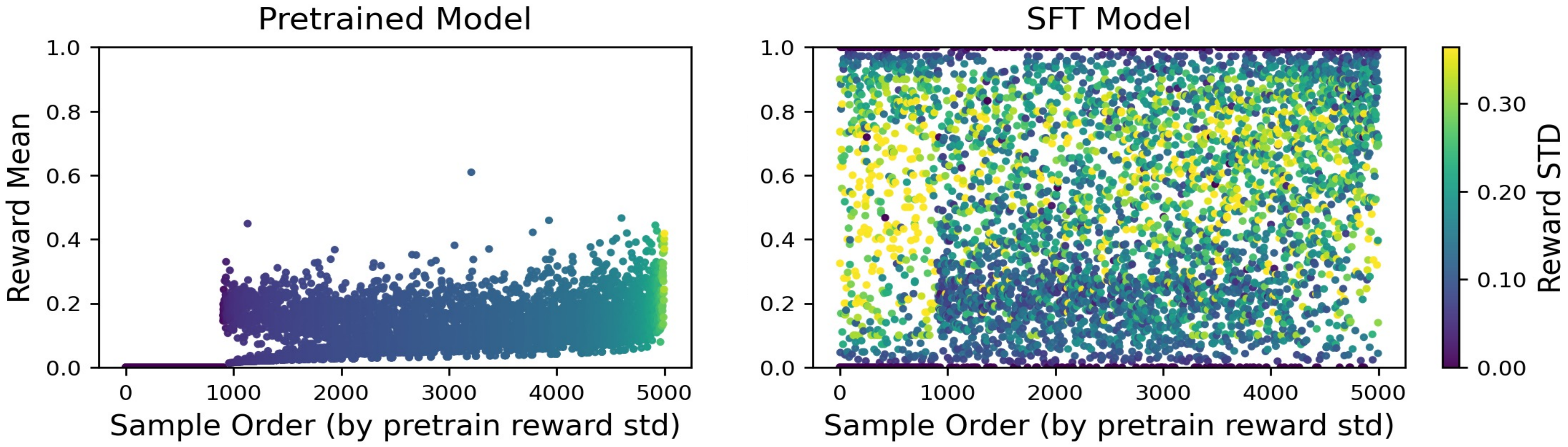

ⓘ **Importance of SFT in RFT pipeline: mitigates vanishing gradients**

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The initial SFT phase does not need to be expensive!

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Reward std is a key quantity to track for successful RFT

Implicit Bias of Policy Gradient in Linear Quadratic Control: Extrapolation to Unseen Initial States

R + Alexander + Cohen-Karlik + Giryes + Globerson + Cohen | *arXiv 2024*

Policy Gradient in Optimal Control

Optimal Control Problem

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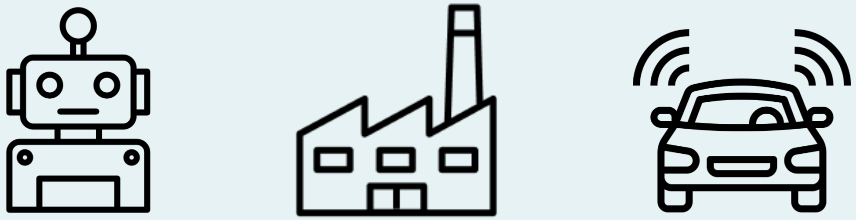
Minimize cost via **gradient descent** w.r.t. controller parameters

Extrapolation to Unseen Initial States

Issue of Prime Importance: Extrapolation to **initial states unseen in training**



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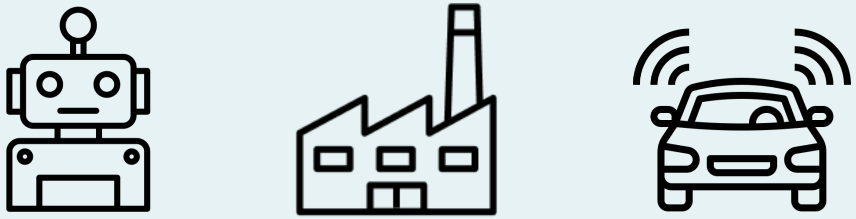

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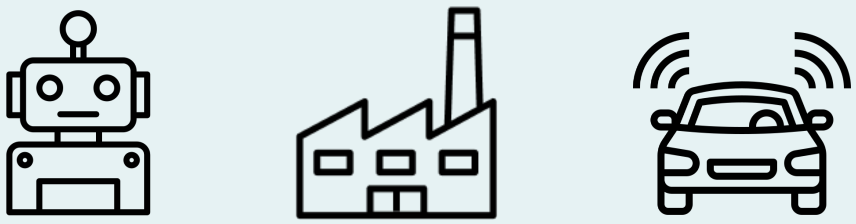

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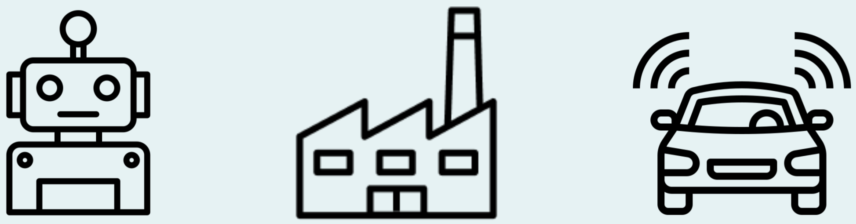

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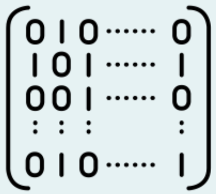

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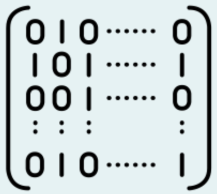

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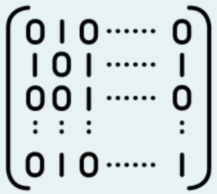

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LQR Problem (state $\mathbf{x}_h \in \mathbb{R}^D$, control $\mathbf{u}_h \in \mathbb{R}^M$)

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Under these assumptions implicit bias is irrelevant

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In this setting the training cost has multiple minimizers

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Controllers minimizing the training cost

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Extrapolation Error

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Quantifying Extrapolation: Baseline Controllers

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Minimizes the training cost but

$\mathcal{E}(\mathbf{K}_{\text{no-ext}})$ is **typically high**

Intuition: Extrapolation Depends on Exploration

Intuition Behind Our Analysis: Extrapolation depends on **degree of exploration induced by the system** from training initial states

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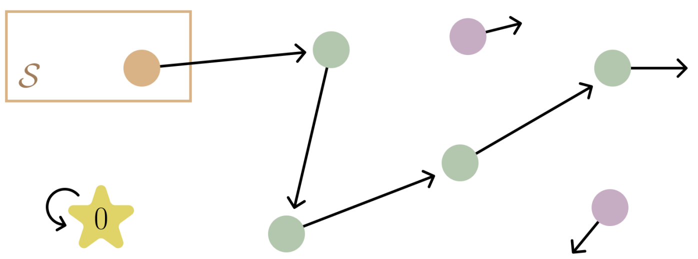
Policy Gradient Iterate t Controller

State Dynamics: $\mathbf{A} + \mathbf{K}^{(t)}$



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- state explored during policy gradient
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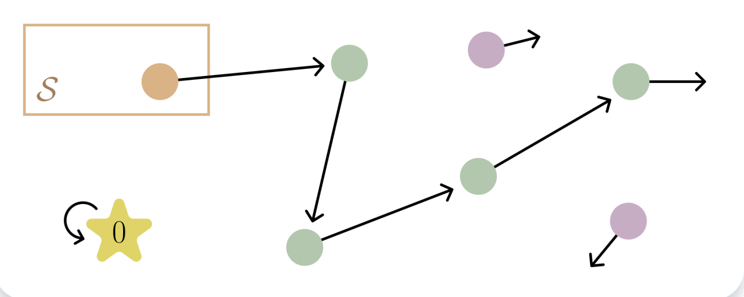

Non-Extrapolating Controller

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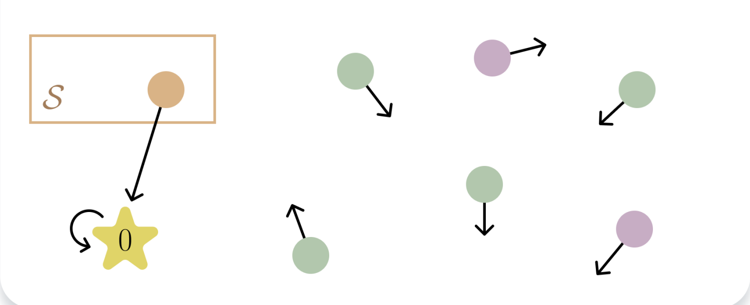

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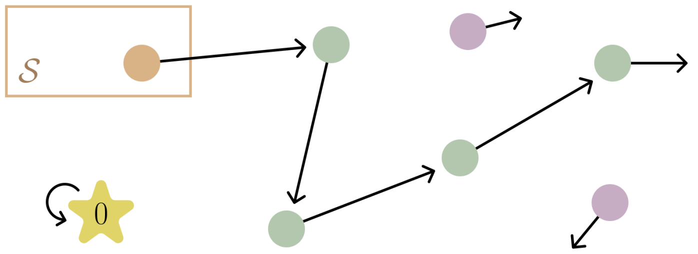

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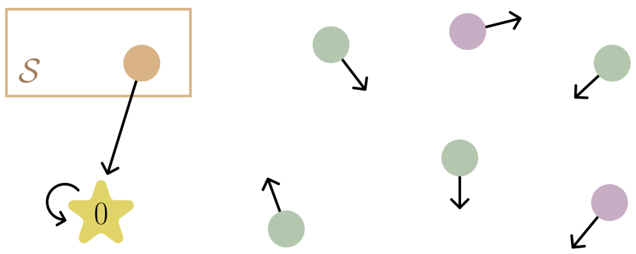
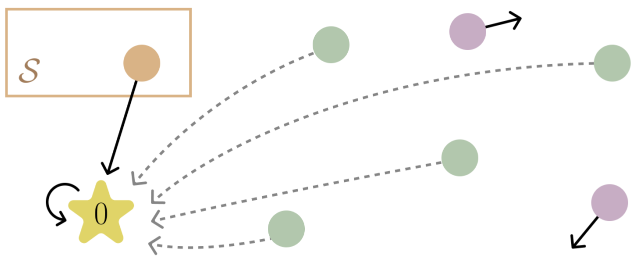


Policy Gradient Final Controller

State Dynamics: $\mathbf{A} + \mathbf{K}_{\text{pg}}$



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$\mathbf{K}^{(t)}$ – the policy gradient controller at iteration t

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- For any $\mathbf{x} \in \mathcal{X}_{\text{pg}}^{\perp}$ the controls produced by $\mathbf{K}^{(t)}$ and $\mathbf{K}_{\text{no-ext}}$ are the same

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- For any $\mathbf{x} \in \mathcal{X}_{\text{pg}}^{\perp}$ the controls produced by $\mathbf{K}^{(t)}$ and $\mathbf{K}_{\text{no-ext}}$ are the same
- There exist systems s.t. $\mathcal{X}_{\text{pg}} \subseteq \operatorname{span}(\mathcal{S})$ and $\mathcal{E}(\mathbf{K}^{(t)}) = \mathcal{E}(\mathbf{K}_{\text{no-ext}})$

Extrapolation in Exploration-Inducing Setting

Q: Exploration is necessary for extrapolation, but can it be sufficient?

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➡

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

Ensures trajectory spans whole state space

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where perfect extrapolation is attained when the horizon $H \to \infty$

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Q: We saw two ends of a spectrum, but which type of extrapolation do we typically get?

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experiments suggest these limitations may be alleviated

Implicit Bias in Optimal Control $\neq$ Euclidean Norm Minimization

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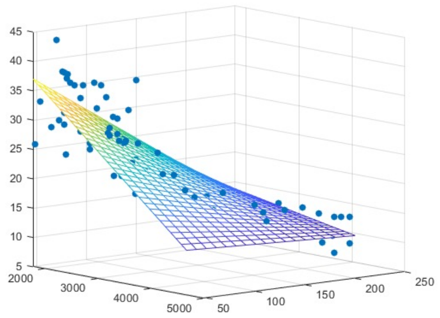
Supervised Learning

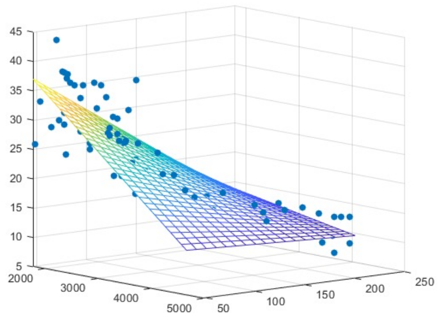


Task: Linear regression

Known (e.g. Zhang et al. 2017)**:** Implicit bias minimizes **Euclidean norm**

Implicit Bias in Optimal Control ≠ Euclidean Norm Minimization

Supervised Learning



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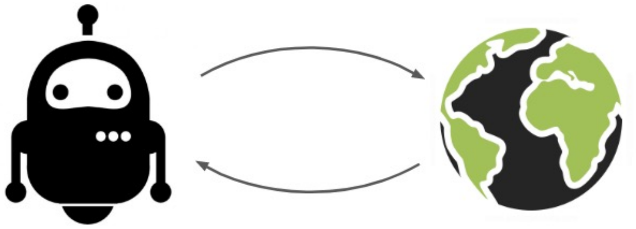
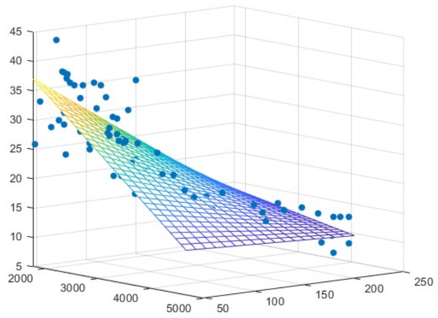


Task: LQR

Implicit Bias in Optimal Control $\neq$ Euclidean Norm Minimization

Supervised Learning



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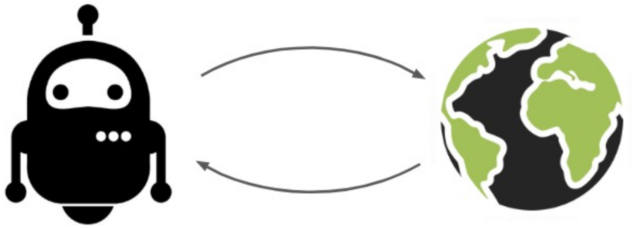

Task: LQR

Our Work: Implicit bias does **not** minimize **Euclidean norm**

Implicit Bias in Optimal Control $\neq$ Euclidean Norm Minimization

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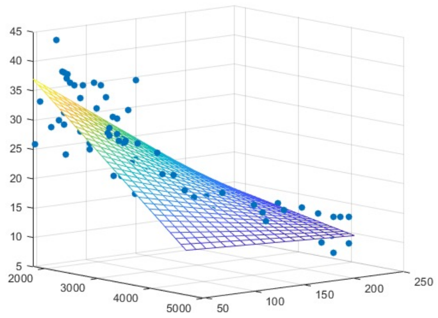

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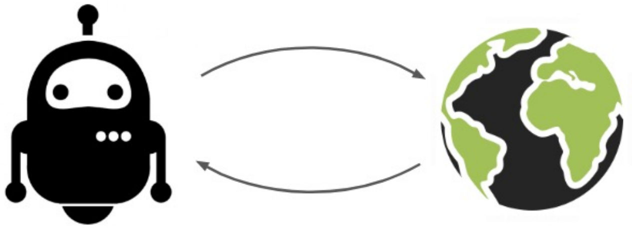

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Among controllers minimizing the training cost, $\mathbf{K}_{\text{no-ext}}$ has the minimal Euclidean norm

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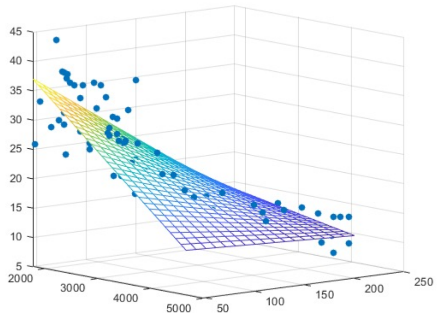

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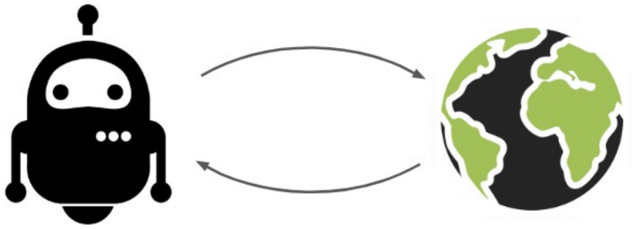

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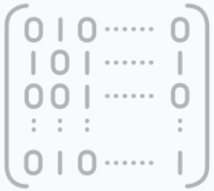
Among controllers minimizing the training cost, $\mathbf{K}_{\text{no-ext}}$ has the minimal Euclidean norm

➡ Extrapolation implies policy gradient **does not implicitly minimize Euclidean norm**

Main Contributions: Effect of Implicit Bias on Extrapolation

Q: To what extent does the implicit bias of policy gradient lead to extrapolation to initial states unseen in training?



Theory for the Linear Quadratic Regulator (LQR) Problem:
Extrapolation depends on an **interplay between the system and initial states seen in training**

Experiments:
Support theory for LQR and demonstrate its conclusions apply to **non-linear systems and neural network controllers**

Experiments: Analyzed LQR Problems

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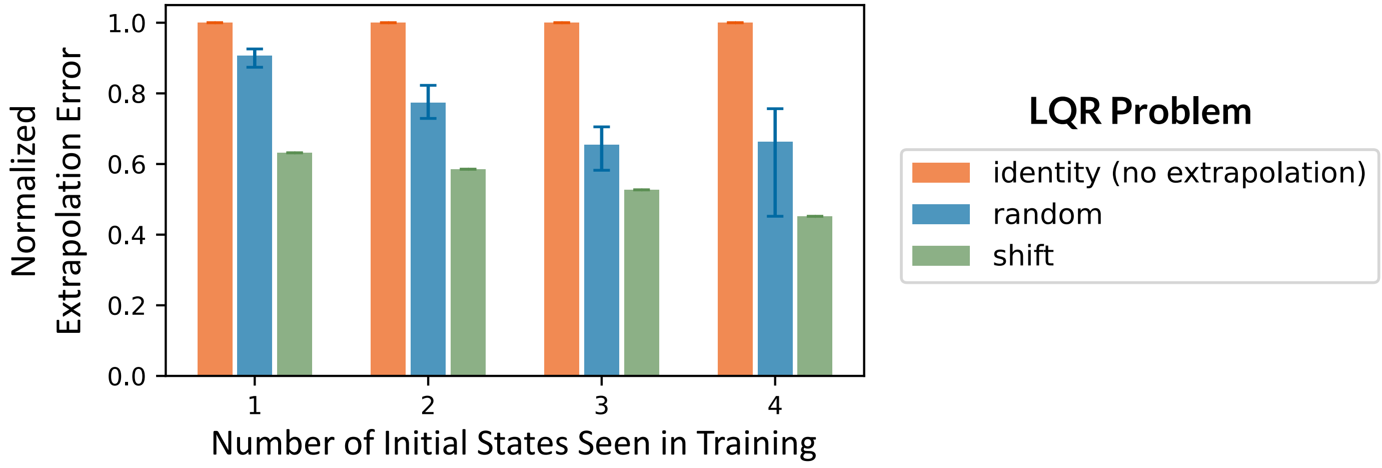
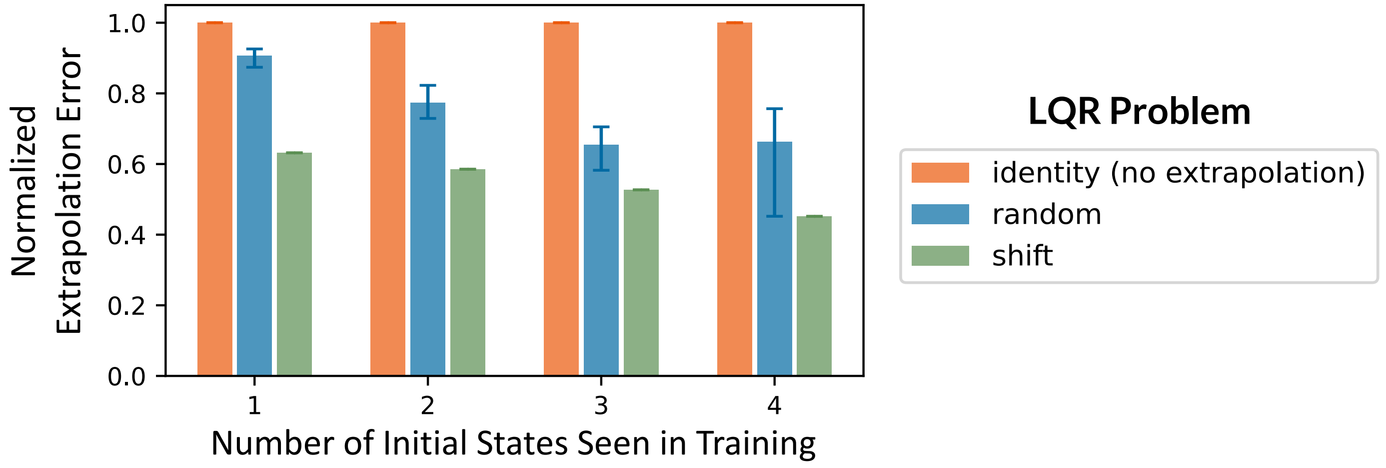


Experiments: Analyzed LQR Problems



In accordance with our theory:

No extrapolation occurs under the identity system, while for the shift and random systems we have non-trivial extrapolation (yet not perfect)

Experiments: Non-Linear Systems and Neural Network Controllers

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Our Theory: Linear system induces exploration
from initial states seen in training

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Pendulum Control Problem

(analogous experiments for a quadcopter control problem)

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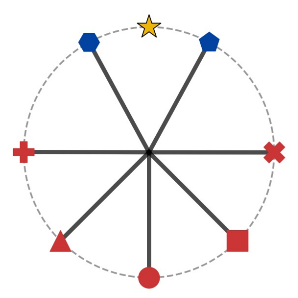
★ target state

● initial state seen in training

● initial state unseen in training

Initial States (time step 0)



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● initial state seen in training
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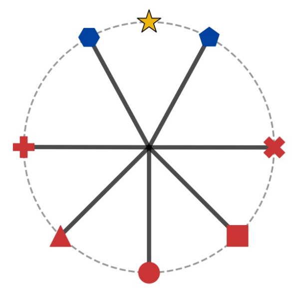
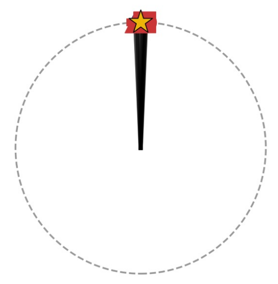


Policy Gradient Controller
Final States (time step 100)



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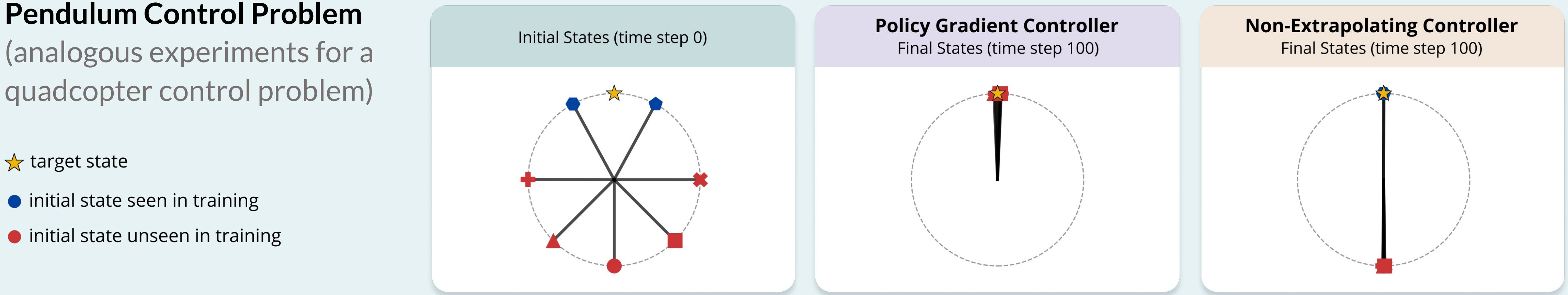

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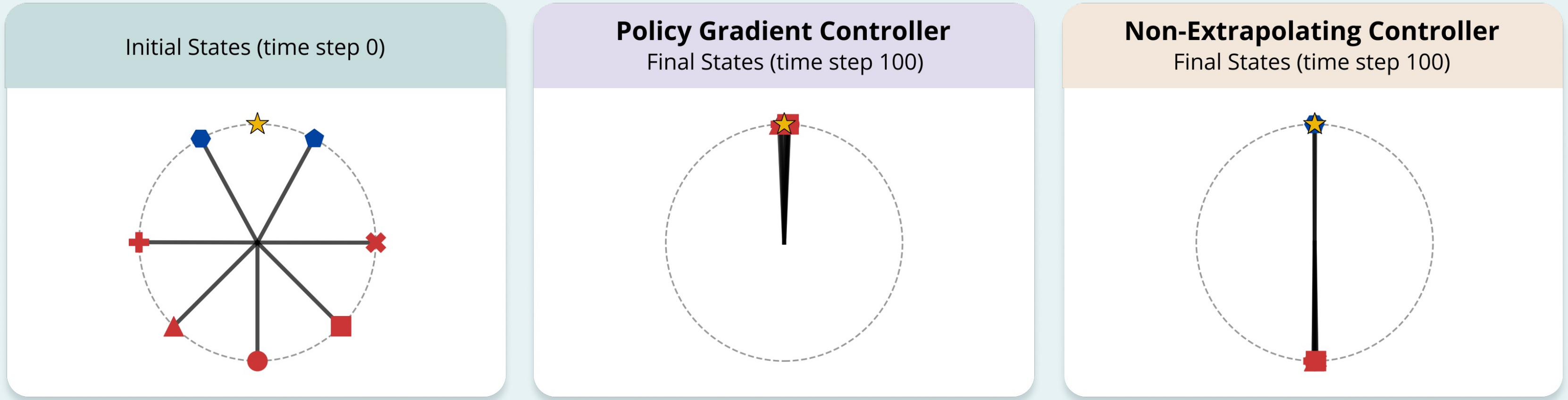
(analogous experiments for a quadcopter control problem)

☆ target state

● initial state seen in training

● initial state unseen in training



ⓘ **The controller learned via policy gradient extrapolates despite existence of non-extrapolating controllers**

Conclusion: Implicit Bias of Policy Gradient in Optimal Control

Q: To what extent does the implicit bias of policy gradient lead to extrapolation to initial states unseen in training?

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Extrapolation depends on exploration induced by the system from initial states seen in training

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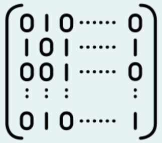

Theory for the LQR Problem: Extrapolation depends on exploration induced by the system from initial states seen in training

Experiments: Support theory for LQR and demonstrate its conclusions apply to non-linear systems and neural network controllers

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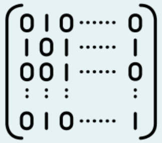

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Going Forward:

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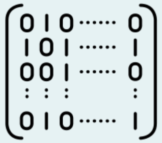

Theory for the LQR Problem: Extrapolation depends on exploration induced by the system from initial states seen in training

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Going Forward:

- Theory for non-linear systems and neural network controllers

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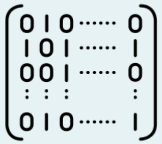


Theory for the LQR Problem: Extrapolation depends on exploration induced by the system from initial states seen in training

Experiments: Support theory for LQR and demonstrate its conclusions apply to non-linear systems and neural network controllers

Going Forward:

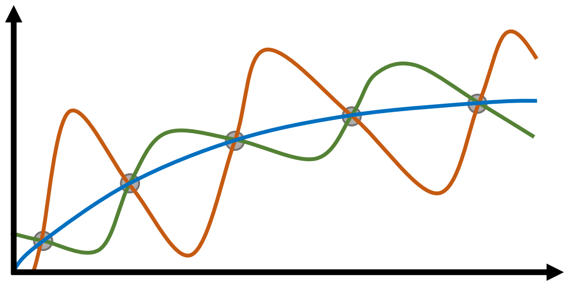
- Theory for non-linear systems and neural network controllers
- Enhancing extrapolation via methods for selecting initial states to train on

Outlook

Optimization and Implicit Bias in Optimal Control/Reinforcement Learning

Optimization and Implicit Bias in Optimal Control/Reinforcement Learning

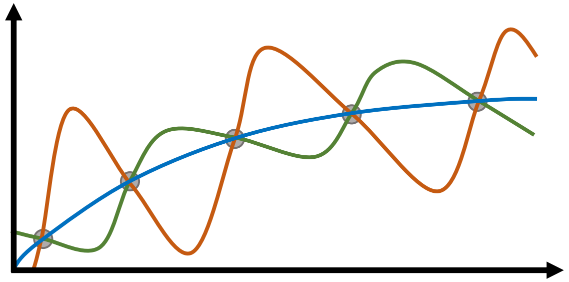
Supervised Learning



Optimization and implicit bias have been extensively studied

Optimization and Implicit Bias in Optimal Control/Reinforcement Learning

Supervised Learning



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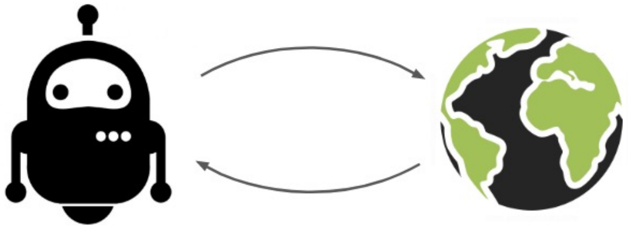
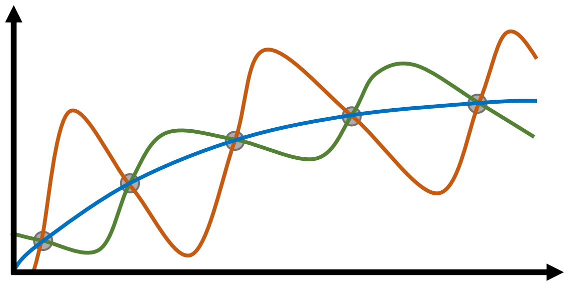
Optimal Control/Reinforcement Learning



Our Results: Optimization and implicit bias can substantially differ from those in supervised learning, hence require dedicated study

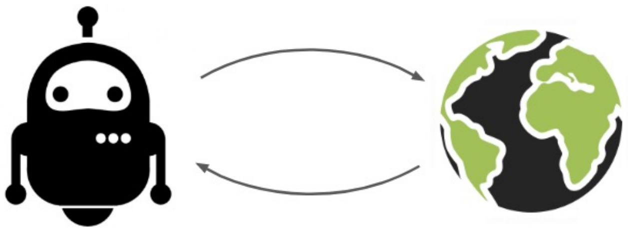
Optimization and Implicit Bias in Optimal Control/Reinforcement Learning

Supervised Learning



Optimization and implicit bias have been extensively studied

Optimal Control/Reinforcement Learning



Our Results: Optimization and implicit bias can substantially differ from those in supervised learning, hence require dedicated study

Studying optimization and implicit bias in optimal control/reinforcement learning may allow addressing their unique challenges

Thank You!

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